

# A representation theorem for upper locally distributive lattices via colored posets

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## Abstract

This paper, proposes a representation theorem for upper locally distributive lattices using colored posets. A colored poset  $P = (X, <, \gamma)$ , is the poset  $P = (X, <)$  equipped with a coloring  $\gamma : X \rightarrow \mathcal{N}$  of its comparability graph. We show that the set of colored ideals of any colored poset, ordered by inclusion is an upper locally distributive lattice, and there is an isomorphism between these lattices and reduced colored posets. We also discuss the relation between upper locally distributive lattices and antimatroids. We close, this paper by giving algorithmic aspects for generating colored ideals and colored linear extensions of colored posets.

**Keywords:** upper locally distributive lattice, closure system, antimatroid, colored poset.

This paper is motivated by representation theory and reconstruction algorithmic aspects for lattices. All objects considered here are supposed to be finite. For definitions on lattices and ordered sets not given here, see Davey and Priestley's book[2].

The main result of this paper, is that, **any upper locally distributive lattice is isomorphic to the lattice of all colored ideals of a colored poset.**

Let us first recall the famous Birkhoff's representation theorem for distributive lattices.

**Theorem 1** [1] *Any distributive lattice  $L$  is isomorphic to the lattice of all order ideals  $\mathcal{I}(J(L))$ , where  $J(L)$  is the poset induced by the set of join-irreducible elements of  $L$ .*

An essential algorithmic aspect of Theorem 1, is the efficient reconstruction of a distributive lattice  $L$  as the lattice of order ideals of the poset  $J(L)$ [5]. For general lattices, several representations have been proposed, but many of them use both join-irreducible and meet-irreducible elements, which are not simple according to Davey and Priestley point of view [2, 8, 10, 11, 19]. The first representation using join-irreducible elements and some essential elements (not necessarily meet-irreducible) was introduced by Wille [18], followed by Duquenne [4] by introducing the join-core of a lattice  $L$  as the smallest partial join-semi-lattice  $E(L)$  such that elements of  $L$  are the order ideals of  $E(L)$  closed by the join operation. Such a representation is still not simply identifiable and no bounds are known about the size of  $E(L)$ .

Dilworth [3] has introduced upper locally distributive lattices and has observed that they are close to distributive lattices. Upper locally distributive lattices have been rediscovered several times, and have many names in the literature (as Antimatroid, Join-distributive lattices, ...). For a survey, see Monjardet [14] which contains many characterizations. In particular, a lattice  $L$  is upper locally distributive if for any  $x \in L$ , the sublattice  $[x, x^*]$  is boolean, where  $x^* = \bigvee \{y \in L \mid y \text{ covers } x\}$  and  $\bigvee$  is the join operation.

This paper proposes a representation for upper locally distributive lattices using colored posets. This confirms Dilworth's observation since posets are just colored posets where all elements have different colors. We show that there is an isomorphism between upper locally distributive lattices and reducible colored posets. We close this paper, by showing that upper locally distributive lattices have efficient generating algorithms similar to those available for distributive lattices [5, 15].

## 1 Colored Posets

**Definition 1** A colored poset, denoted by  $P = (X, <, \gamma)$ , is the poset  $P = (X, <)$  equipped with a coloring  $\gamma : X \rightarrow \mathcal{N}$  such that  $x < y$  implies  $\gamma(x) \neq \gamma(y)$ . In others words,  $\gamma$  is a coloring of the comparability graph of  $P$ .

Let  $P = (X, <, \gamma)$  be a colored poset and  $I$  a subset of  $X$ .  $I$  is said an order ideal of  $P$  (ideal for short) if  $x \in I$  and  $y < x$  implies  $y \in I$ . We define the *colored ideal*  $\mathcal{C}(I) = \{i \in \mathcal{N} \mid \exists x \in I, \gamma(x) = i\}$  as the set of colors in  $I$ . Note that two different ideals may have the same set of colors as shown in Figure 1; i.e.  $\mathcal{C}(\{a, b, c\}) = \mathcal{C}(\{a, b, d\}) = \mathcal{C}(\{a, b, c, d\}) = \{123\}$ .

The set of all colored ideals of  $P$ , denoted by  $\mathcal{C}(P)$ , is a union closure system containing the empty set, and hence a lattice; called the lattice of all colored ideals of  $P$ .

**Remark 1** The set of colored ideals of a colored poset is not always an intersection closure system, otherwise it would be a distributive lattice. Figure 1 shows that  $\{23\} \cap \{13\} = \{3\} \notin \mathcal{C}(P)$ .

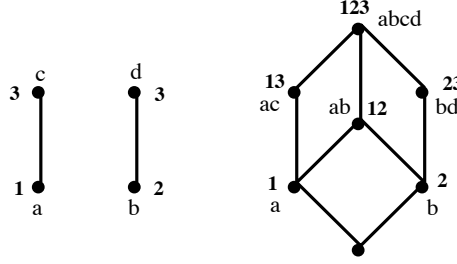


Figure 1: An example of a colored poset and its lattice of colored ideals.

We define the mapping  $gen : \mathcal{C}(P) \rightarrow \mathcal{I}(P)$  by  $gen(C) = \{x \in X \mid \gamma(\downarrow x) \subseteq C\}$  (i.e. the greatest ideal  $I$  of  $P$  such that  $\gamma(I) = C$ ) where  $\gamma(\downarrow x) = \{\gamma(y) \mid y \leq x\}$  and  $\mathcal{I}(P)$  the set of ideals of  $P$ . Clearly, for any  $C \in \mathcal{C}(P)$  and a minimal element  $x \in P \setminus gen(C)$ , we have  $\gamma(x) \notin C$ . Note that the mapping  $gen$  is an order embedding of  $\mathcal{C}(P)$  into  $\mathcal{I}(P)$ .

**Proposition 1** *Let  $P = (X, <, \gamma)$  be a colored poset. Then  $\mathcal{C}(P) = (\mathcal{C}(P), \subseteq)$  is an upper locally distributive lattice.*

**Proof:** To show that  $\mathcal{C}(P)$  is an upper locally distributive, we need to proof that for any  $C \in \mathcal{C}(P)$  the set of all colored ideals covering  $C$  generates a boolean lattice [3].

Let  $C \in \mathcal{C}(P)$  and  $K = \{\gamma(x) \mid x \text{ is minimal in } P \setminus gen(C)\}$ . We note that for any  $c \in K$ ,  $c \notin C$  by maximality of  $gen(C)$ , and therefore the set  $\{C \cup \{c\} \mid c \in K\}$  are the only covering elements of  $C$ . So, for each subset  $A \in 2^K$ ,  $C \cup A$  is a colored ideal of  $P$ . Thus the interval  $[C, C \cup K]$  is a boolean sublattice.  $\square$

To show the isomorphism between the lattice of all colored ideals of colored poset and an upper locally distributive lattice, we need to recall some properties of lattices.

For a lattice  $L$  lattice, we denote by  $Bip(L) = (J(L), M(L), \preceq_L)$  the representation of  $L$  by its irreducible bipartite poset. Markowsky [11, 12] showed that the union closed system  $\mathcal{F}(L) = \{A^\cup \mid A \subseteq J(L)\}$  ordered by inclusion is isomorphic to  $L$ , where  $A^\cup = \{m \in M(L) \mid (\exists j \in A) j \preceq_L m\}$ .

**Property 1** [13] *Let  $L = (X, <_L)$  be an upper locally distributive lattice. For any  $A, B \in \mathcal{F}(L)$ , if  $B$  covers  $A$  then  $|B \setminus A| = 1$ .*

Let  $L = (X, <_L)$  be an upper locally distributive lattice. We define the poset

$P(L) = (J(L), <_L, \gamma)$  by the induced poset of join-irreducible elements  $J(L) = (J(L), <_L)$  and a coloring  $\gamma : J(L) \rightarrow M(L)$  with  $\gamma(j) = \{j\}^\cup \setminus \bigcup_{j' <_L j} \{j'\}^\cup$ . Now, let us show that  $P(L)$  is a colored poset.

**Lemma 1** *Let  $L = (X, <_L)$  be an upper locally distributive lattice. Then  $P(L) = (J(L), <_L, \gamma)$  is a colored poset.*

**Proof:** It suffices to show that each element  $j \in P(L)$  has exactly one color, since by definition of  $\gamma$  two comparable elements have different colors.

We know, that for any lattice, a join-irreducible element  $j$  covers  $j_*$  with  $j_* = \bigvee \{j' \in J(L) \mid j' <_L j\}$  corresponding to the set  $\bigcup_{j' <_L j} \{j'\}^U$  in  $\mathcal{F}(L)$ . Using Property 1, we have  $|\{j\}^U \setminus \bigcup_{j' <_L j} \{j'\}^U| = 1$  and therefore  $|\gamma(j)| = 1$ .  $\square$

**Main Theorem 1** *Any upper locally distributive lattice  $L$  is isomorphic to the lattice of all colored ideals  $\mathcal{C}(P(L))$ .*

**Proof:** It suffices to show that  $\mathcal{C}(P(L)) = \mathcal{F}(L)$ .

Let  $C \in \mathcal{C}(P(L))$ . Then  $I = \text{gen}(C)$  is an ideal of  $J(L)$  and by definition of  $\mathcal{F}(L)$   $I^U \in \mathcal{F}_L$ .

Now let  $A \in \mathcal{F}_L$ . Then there exists a set  $I \subseteq J(L)$  with  $I^U = A$ . Suppose that  $I$  is maximal with this property, thus  $I$  is an ideal of  $J(L)$  and therefore  $\gamma(I) = I^U = A \in \mathcal{C}(P(L))$ .  $\square$

Clearly, we cannot conclude as in the distributive case, that there is an isomorphism between colored posets and upper locally distributive lattices. Since two non-isomorphic posets can have the same lattice of colored ideals (see Figure 2). To establish a bijection, the reduction of a colored poset has to be introduced, using the characterization of the poset of irreducible elements [11].

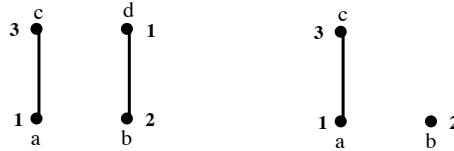


Figure 2: A colored poset and its reduction

**Definition 2** *Let  $P = (X, <, \gamma)$  be a colored poset.  $P$  is said to be reduced if for any  $x \in X$  there is no ideal  $I \subseteq X \setminus \{x\}$  such that  $\gamma(I) = \gamma(\downarrow x)$ .*

**Proposition 2** [15] *There is an isomorphism between lattices and irreducible bipartite posets.*

Two colored posets  $P = (J, <_P, \gamma)$  and  $Q = (K, <_Q, \beta)$  are said to be isomorphic if and only if there is an isomorphism  $\psi$  from  $(J, <_P)$  onto  $(K, <_Q)$  and for  $x, y \in J$ ,  $\gamma(x) = \gamma(y)$  iff  $\beta(\psi(x)) = \beta(\psi(y))$ .

Now the isomorphism follows immediately from Proposition 2, since irreducible bipartite posets for upper locally distributive lattices are isomorphic to reduced colored posets.

**Corollary 1** *There is an isomorphism between upper locally distributive lattices and reduced colored posets.*

The following theorem relates antimatroid and previous results. For more details, on matroids and antimatroids, see [9].

**Definition 3** *A union closure system  $\mathcal{F}$  is an antimatroid iff for every  $A \neq \emptyset \in \mathcal{F}$  there exists  $m \in A$  such that  $A \setminus \{m\} \in \mathcal{F}$ .*

**Theorem 2** *Let  $\mathcal{F}$  be a union closure system. Then the following statements are equivalents:*

1.  $\mathcal{F}$  is an antimatroid.
2.  $L = (\mathcal{F}, \subseteq)$  is upper locally distributive lattice.
3.  $\mathcal{F}$  is the set of colored ideals of the colored poset  $P(L)$ .
4. For any  $A, B \in \mathcal{F}$ , if  $B$  covers  $A$  then  $|B \setminus A| = 1$ .

## 2 Algorithmic aspects of colored posets

This section discuss algorithms for generating colored ideals and colored linear extensions of a colored poset.

### 2.1 Generating colored ideals

Let  $P = (J, <, \gamma)$  be a colored poset. Our strategy to generate colored ideals of a colored poset is the same as generating classical ideals of a poset [5]. Namely, we choose a minimal  $j \in J$ . We first generate the colored ideals containing  $\gamma(j)$ , and we backtrack to generate those not containing  $\gamma(j)$ . This idea is formulated in the following proposition:

**Proposition 3** *Let  $P = (J, <, \gamma)$  be a colored poset,  $C$  a colored ideal of  $P$  and  $j$  a minimal element in  $P \setminus \text{gen}(C)$ . Then*

1.  $\gamma(j) \notin C$ .

2.  $gen(C \cup \gamma(j)) = gen(C) \cup \{j' \in J \mid \gamma(\downarrow j') \subseteq C \cup \gamma(j)\}$ .
3. The colored ideals containing  $C \cup \gamma(j)$  are the colored ideals of the restriction of  $P$  to  $J \setminus gen(C \cup \gamma(j))$ .
4. The colored ideals not containing  $C \cup \gamma(j)$  are the colored ideals of the restriction of  $P$  to  $J \setminus \bigcup_{j' \in J \mid \gamma(j') = \gamma(j)} \uparrow j'$ .

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**Algorithm 1:** COLORIDEAL( $C, I, J$ )

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begin
  if  $J = \emptyset$  then
1    Process( $(C, I)$ );
  else
2     $j =$  choose a minimal element of  $J$ ;
3    COLORIDEAL( $C \cup \gamma(j), gen(C \cup \gamma(j)), J \setminus gen(C \cup \gamma(j))$ ); the colored
    ideals containing  $\{C \cup \gamma(j)\}$ 
4    COLORIDEAL( $C, I, J \setminus \bigcup_{\{j' \in J \mid \gamma(j') \in C \cup \gamma(j)\}} \uparrow j'$ ); the colored ideals not
    containing  $\{C \cup \gamma(j)\}$ 
  end
end

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**Theorem 3** Algorithm 1 generate the set of all colored ideals of  $P$  using  $O(\Delta_c \cdot |\mathcal{C}(P)|)$  time complexity, with  $\Delta_c$  the maximal value of  $\sum_{\{j \in J \mid \gamma(j) = c\}} \Delta(j)$  for any color  $c \in P$ , where  $\Delta(j)$  is the maximum in-degree of  $j$  in  $P$ .

**Proof:**

The correctness of Algorithm 1 follows from Proposition 3.

For each call of the algorithm  $COLORIDEAL(C, I, J)$  :

- We compute  $gen(C \cup \gamma(j))$  and delete them from  $J$ . To do so, we need only to compute the set  $I' = \{j' \in J \mid \gamma(\downarrow j') \subseteq C \cup \gamma(j)\}$ . Clearly, if  $x \in I'$  and  $y \leq x$  then  $y \in I'$ . Thus we can compute  $I'$  by simply deleting minimal elements. The time complexity needed to delete  $I'$  can be charged on previous colored ideals contained in  $C \cup \gamma(j)$ , since any color of  $C \cup \gamma(j)$  has given a new colored ideal. Thus the time needed for one colored ideal (generated by a color  $c$ ) is bounded by  $\sum_{\{j \in J \mid \gamma(j) = c\}} \Delta(j)$ .
- We compute the set  $I'' = \{j' \mid j' \in J, \gamma(j') = \gamma(j)\}$  and delete them and their successor from  $J$ . To locate the set  $I''$  can be done in  $O(|\{j' \in J \mid \gamma(j') = \gamma(j)\}|)$

using a lookup table. Deleting the elements of  $P'$  from  $J$  can be done by using a counter of each element of  $J$  initialized to the number of its predecessors. The counter of an element become 0 when the element is minimal, and  $\infty$  when is deleted. This is a classical technique used for generating ideals for poset [5].

Thus the amortized time complexity per ideal is bounded by  $\Delta_c$ .  $\square$

For Gray code generation of colored ideals, Priesse and Ruskey [17] have shown that the elements of an antimatroid can be generated in a Gray code manner, such that two successive elements differ by at most 2. Hence, according to Theorem 2, their algorithm can be applied for generating colored ideals of a colored poset.

**Proposition 4** [17] *Let  $P = (J, <, \gamma)$  be a colored poset. Then there exists a sequence  $S = (\emptyset = C_1, C_2, \dots, C_{|C(P)|})$  for colored ideals of  $P$  such that  $|C_i \setminus C_{i-1}| \leq 2$ .*

**Remark 2** *Generating colored ideals in lexicographic order (as for ideals), by simulating a covering tree rooted at the bottom is not always possible [6]. Figure 3 shows a counter-example.*

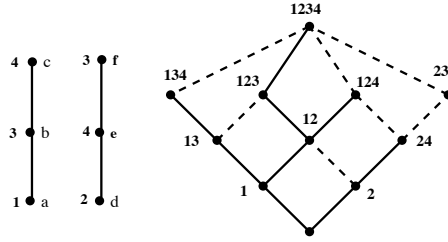


Figure 3: The colored ideal 234 cannot be reached from the bottom in a lexicographic order.

## 2.2 Generating colored linear extensions

Let  $P = (X, <, \gamma)$  be a colored poset and  $\sigma = x_1, x_2, \dots, x_n$  a linear extension of  $P$ . We denote by  $First(c)$  the first element in  $\sigma$  such that  $\gamma(x) = c$ . For each linear extension  $\sigma$ , we associate a colored linear extension  $\mathcal{C}(\sigma) = c_1, c_2, \dots, c_m$  ( $m \leq n$ ) of  $P$  such that  $c_1 <_{\mathcal{C}(\sigma)} c_2$  iff  $First(c_1) <_{\sigma} First(c_2)$ .

**Corollary 2** *Colored linear extensions of  $P$  are in bijection with maximal chains of the colored ideals lattice.*

**Proposition 5** [17] *There exists a combinatorial Gray code for generating colored linear extensions of a colored poset.*

### 3 Conclusion

We have introduced the notion of a colored poset which is a generalization of a classical poset. The notion of colored ideals lattice has allowed us to generalize Birkhoff's representation theorem for distributive lattices, and then inherit its algorithmic aspects.

The generalization of colored posets to set-colored posets for general lattices seems to have nice properties. Indeed, many classes of lattices can be characterized by their associated colored poset. Moreover, we think that representations by colored posets can be used as a kind of distributivity measure of a lattice. The following proposition shows such an example:

**Proposition 6** *Let  $L$  be an upper locally distributive lattice. If the poset  $J(L) = (J(L), <_L)$  is an interval order then  $L$  is distributive.*

**Proof:** Let  $P(L) = (J(L), <_L, \gamma)$  be the associated colored poset of  $L$ . It suffices to show that all elements have different colors.

Since,  $(J(P), <_L)$  is an interval order, then there exists a sequence  $j_1, j_2, \dots, j_n$  of  $P(L)$  such that  $\downarrow j_1 \setminus \{j_1\} \subseteq \downarrow j_2 \setminus \{j_2\} \subseteq \dots \subseteq \downarrow j_n \setminus \{j_n\}$ . Now, suppose that  $\gamma(j_i) = \gamma(x_k)$  for  $i < k$ . Then  $\gamma(\downarrow j_k) = \gamma(\downarrow j_k \setminus \{j_k\}) \cup \gamma(\downarrow j_i)$  since  $\downarrow j_i \setminus \{j_i\} \subseteq \downarrow j_k$ . Thus  $P(L)$  is not reducible which is a contradiction.  $\square$

This generalization and opens problems can be found in [7, 16].

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